

Multi-latin squares

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Abstract

A multi-latin square of order n and index k is an $n \times n$ array of multi-sets, each of cardinality k , such that each symbol from a fixed set of size n occurs k times in each row and k times in each column. A multi-latin square of index k is also referred to as a k -latin square. A 1-latin square is equivalent to a latin square, so a multi-latin square can be thought of as a generalization of a latin square.

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In this note we show that any partially filled-in k -latin square of order m embeds in a k -latin square of order n , for each $n \geq 2m$, thus generalizing Evans' Theorem. Exploiting this result, we show that there exist non-separable k -latin squares of order n for each $n \geq k + 2$. We also show that for each $n \geq 1$, there exists some finite value $g(n)$ such that for all $k \geq g(n)$, every k -latin square of order n is separable.

We discuss the connection between k -latin squares and related combinatorial objects such as orthogonal arrays, latin parallelepipeds, semi-latin squares and k -latin trades. We also enumerate and classify k -latin squares of small orders.

Keywords: latin square, multi-latin square, orthogonal array, semi-latin square, SOMA, latin parallelepiped.

1 Introduction

For each positive integer a , we use the notation $N(a)$ for the set of positive integers $\{1, 2, \dots, a\}$. The operations and relations used in this paper always take into account the multiplicity of symbols in multisets. For example, if A and B are multisets, and A contains t_1 occurrences of symbol x and B contains t_2 occurrences of symbol x , then $A \cup B$ contains $t_1 + t_2$ occurrences of symbol x . Similarly, $A \subseteq B$ if and only if for each symbol $x \in A$ that occurs t_1 times in A and t_2 times in B , $t_1 \leq t_2$.

A *partial k -latin square* of order n is an $n \times n$ array, where each cell of the array contains a multiset of cardinality at most k with symbols from $N(n)$, such that each symbol occurs at most k times in each row and at most k times in each column. A *k -latin square* is a partial k -latin square in which each cell contains exactly k symbols, and hence each symbol occurs precisely k times in each row and k times in each column. We sometimes refer to (partial) k -latin squares as *(partial) multi-latin squares*, of *index k* .

Below is a 2-latin square of order 4:

1, 2	1, 2	3, 3	4, 4
2, 3	2, 3	4, 4	1, 1
1, 4	3, 4	1, 2	2, 3
3, 4	1, 4	1, 2	2, 3

Note that a (partial) 1-latin square is equivalent to a (partial) latin square. Thus multi-latin squares may be thought of as generalizations of latin squares.

A (partial) k -latin square is said to be *simple* if each cell contains no repeated symbols. So multisets are forbidden in simple partial k -latin squares.

For a (partial) k -latin square L of order n , and for any $i, j \in N(n)$, write $L(i, j) = A$ whenever cell (i, j) of L contains the multiset A . We may thus consider a (partial) k -latin square as a set of ordered triples of the form $(i, j, L(i, j))$,

where in some cases $L(i, j)$ may be equal to the empty set.

In this sense, a (partial) k -latin square L of order n can be thought of as a subset of a (partial) k -latin square L' of order n if and only if for each $i, j \in N(n)$, $L(i, j) \subseteq L'(i, j)$.

We begin with some straightforward existence lemmas.

Lemma 1.1. *For all positive integers n and k , there exists a k -latin square of order n .*

Proof. Let L be a latin square of order n . Let L_k be the k -latin square of order n , where for each $i, j \in N(n)$, $L_k(i, j)$ is the multiset consisting of k copies of the symbol in cell (i, j) of L . \square

Throughout this paper, for any integer x and positive integer n , we define $(x \bmod n)$ to be the unique member of $N(n)$ which is congruent to x modulo n .

Lemma 1.2. *For all positive integers n and k , there exists a simple k -latin square of order n if and only if $n \geq k$.*

Proof. If $n < k$, then we are forced to have at least one repeated symbol in a cell, contradicting the simple criterion. For each $n \geq k$, we define a simple k -latin square L_k of order n as follows. For each $i, j \in N(n)$,

$$L_k(i, j) = \{(i + j \bmod n), (i + j + 1 \bmod n), \dots, (i + j + k - 1 \bmod n)\}.$$

\square

We explore the relationship between multi-latin squares and other combinatorial configurations in Section 2. In Section 3, we examine whether standard embedding theorems on latin squares generalize to multi-latin squares. In Section 4, we explore when multi-latin squares “separate” into other multi-latin squares. Making use of the embedding results from Section 3, we show the existence of non-separable k -latin squares of order n for each $n \geq k + 2$. Then in Section 5 we count and classify k -latin squares of order n for small values of k and n . Finally in Section 6 we mention some possible applications of multi-latin squares to experimental design.

2 Equivalences and Connections

Multi-latin squares are equivalent to at least three other types of combinatorial objects, and are related to many more.

There is an equivalence between multi-latin squares and certain *orthogonal arrays*.

Definition 2.1. An orthogonal array of size N , with m constraints, q levels, strength t , and index λ , denoted by $\text{OA}_\lambda(N, m, q, t)$ is an $m \times N$ array with entries from \mathbb{Z}_q , such that in every $t \times N$ subarray, every tuple in \mathbb{Z}_q^t appears exactly $\lambda = N/q^t$ times.

Observe that a k -latin square of order n is equivalent to an orthogonal array $\text{OA}_k(kn^2, 3, n, 2)$.

The research on orthogonal arrays tends to deal with the existence of orthogonal arrays with particular parameters and properties [8]. As we saw in the previous section, it is trivial to verify the existence of k -latin squares of order n and hence the existence of an $\text{OA}_k(kn^2, 3, n, 2)$, for each positive k and n .

We can also define multi-latin squares in terms of graph decompositions. It is well-known that a latin square is equivalent to the decomposition of the edges of a complete tripartite graph $K_{n,n,n}$ into triangles. This equivalence has a natural extension to multi-latin squares. We define $kK_{n,n,n}$ to be the multi-graph obtained by replacing each edge of $K_{n,n,n}$ with k parallel edges. Then a k -latin square of order n is equivalent to a decomposition of the edges of $kK_{n,n,n}$ into triangles.

From this equivalence, it is immediate that any graph automorphism of $K_{n,n,n}$ may be applied to a k -latin square L_1 of order n to obtain another k -latin square L_2 of order n which is combinatorially equivalent to L_1 . We say that L_1 and L_2 with these properties are *paratopic* or belong to the same *main class* or *species*. This extends, in a natural way, existing definitions which are used to describe latin squares.

Anderson and Hilton [1] define an (*exact*) (p, q, x) -latin rectangle to be a rectangular matrix with x symbols in each cell such that each symbol occurs at most (*exactly*) p times in each row and at most (*exactly*) q times in each column. Thus a k -latin square is equivalent to an exact (k, k, k) -latin rectangle. Some embedding results for (p, q, x) -latin rectangles are given in [1].

Next we discuss non-equivalent but related objects to multi-latin squares. A $(n \times n)/k$ *semi-latin* square is an $n \times n$ array A , whose entries are k -subsets of a $N(kn)$, such that each element of $N(kn)$ occurs exactly once in each row and once in each column of A . (These objects are referred to as $n \times n$ r -multi latin squares in [14, 15]). Semi-latin squares have been studied extensively, particularly in terms of experimental design. A summary of results on semi-latin squares may be found at [3]; a list of enumerative results in [5]. An embedding theorem for semi-latin squares is given in [15]. If no pair of letters occurs more than once in each cell, the semi-latin square is called *simple*, and is equivalent to a SOMA or *Simple Orthogonal Multi-Array* (see [2], [21], [22]).

Clearly a $(n \times n)/k$ semi-latin square is a distinct concept to a k -latin square of order n . However there are some connections between these two combinatorial objects. Let f be a function $f : N(kn) \rightarrow N(n)$ such that each element of $N(n)$ has a pre-image of size k . Then, f , when applied to the symbol set, maps any

$(n \times n)/k$ semi-latin square to a k -latin square of order n .

In fact, there is a reverse process. Let L be a k -latin square of order n . Given a symbol $x \in N(n)$, construct a bipartite multi-graph B_x with t edges between vertices r_i and c_j if and only if x occurs precisely t times in cell $L(i, j)$. Evans' theorem and the rules of a k -latin square guarantee that the edges of B_x partition into k pairwise-disjoint perfect matchings. Thus, given any inverse $f^{-1} : N(n) \rightarrow N(kn)$ to the function f above, we obtain a $(n \times n)/k$ semi-latin square. (Of course f^{-1} is not a well-defined function.)

However, this relation between multi-latin squares and semi-latin squares is not, in general, one of correspondence. For example, consider the following $(3 \times 3)/2$ semi-latin square:

1, 2	3, 4	5, 6
3, 5	6, 1	4, 2
4, 6	5, 2	3, 1

The maps $f_1, f_2 : N(6) \rightarrow N(3)$ defined by $f_1(1) = f_1(2) = 1, f_1(3) = f_1(4) = 2, f_1(5) = f_1(6) = 3, f_2(1) = f_2(6) = 1, f_2(2) = f_2(3) = 2, f_2(4) = f_2(5) = 3$ give rise to 2-latin squares of order 3, L and L' :

1, 1	2, 2	3, 3
2, 3	1, 3	1, 2
2, 3	1, 3	1, 2

L

1, 2	2, 3	3, 1
2, 3	1, 1	2, 3
3, 1	2, 3	1, 2

L'

The 2-latin squares L and L' are not paratopic.

Just as the differences between latin squares are defined by *latin trades* [6], the differences between multi-latin squares gives rise to a type of combinatorial trade. Such trades would include the $t - (v, k)$ latin trades recently introduced in [11] and [16].

It would be an intriguing line of research to further explore the relationship between multi-latin squares, semi-latin squares and SOMAs.

3 Embeddings

For $1 \leq m \leq n$, we define an $m \times n$ k -latin rectangle to be a partial k -latin square of order n , of which m rows are filled and the remaining rows are empty. Evans' theorem tells us that any partial latin rectangle may be extended to a latin square [7]; this result can be generalized to k -latin rectangles as follows.

Lemma 3.1. *Any $m \times n$ k -latin rectangle embeds in a k -latin square of order n .*

Proof. Let P be an $m \times n$ k -latin rectangle. For each $i \in N(n)$, let $D_0(i)$ be the multiset containing $k - z_i(e)$ copies of symbol e , $1 \leq e \leq n$, where $z_i(e)$ is the number of occurrences of e in column i of P . Observe that each $D_0(i)$ has size $k(n - m)$, and each symbol e occurs $k(n - m)$ times in sets of the form $D_0(i)$.

We now iteratively define multisets $D_x(1), \dots, D_x(n)$, for each x such that $1 \leq x < k(n - m)$. We assume that for a given p , where $0 \leq p < k(n - m)$, we can choose a symbol from each of the multisets $D_p(1), \dots, D_p(n)$, so that the n symbols chosen are all distinct. We then remove the selected symbols to obtain $D_{p+1}(1), \dots, D_{p+1}(n - m)$. For each $i \in N(n)$, we add the selected symbol from $D_p(i)$ to cell $(m + 1 + \lfloor p/k \rfloor, i)$ of P ; in this way we obtain a completion of P to a k -latin square.

We now justify our inductive assumption. By Hall's Theorem, it suffices to show that the union of any s -subset of $\{D_p(1), \dots, D_p(n)\}$ contains at least s distinct symbols. Recall that each of the symbols from $N(n)$ occurs a total of $k(n - m)$ times in the multisets $\{D_0(1), \dots, D_0(n)\}$ and that each of these multisets has cardinality $k(n - m)$. At each stage of the iteration we remove one copy of each of the n symbols (one element from each multiset), hence by induction each of the n symbols will occur a total of $k(n - m) - p$ times in the multisets $\{D_p(1), \dots, D_p(n)\}$ and each of these multisets will have cardinality $k(n - m) - p$. Thus any s -subset of these multisets will contain a total of $s(k(n - m) - p)$ elements, with each symbol occurring at most $k(n - m) - p$ times, and so must contain at least s distinct symbols. \square

If we restrict ourselves to simple k -latin squares, we do not have a direct equivalent of the above lemma. For example, the following 2×3 2-latin rectangle has no completion to a simple 2-latin square of order 3.

1, 2	1, 3	2, 3
1, 2	1, 3	2, 3

It remains an open problem to determine under what conditions a simple multi-latin rectangle can be extended to a simple multi-latin square.

Theorem 3.1. *Any partial k -latin square P of order m embeds in a k -latin square of order n , for each $n \geq 2m$.*

Proof. Let P_0 be the partial k -latin square of order n defined by $P_0(i, j) = \emptyset$, if $i > m$ or $j > m$, and otherwise

$$\begin{aligned} P_0(i, j) &= P(i, j) \cup \{(i + j \bmod m) + m, (i + j + 1 \bmod m) + m, \\ &\quad \dots, (i + j + k - |P(i, j)| - 1 \bmod m) + m\}. \end{aligned}$$

We now wish to complete rows 1 through to m . For each $i \in N(m)$, let $X(i)$ be the multiset $\bigcup_{j=1}^m P_0(i, j)$. Define $A_0(i)$ to be the multiset containing $k - x_i(e)$ copies of symbol e , $1 \leq e \leq n$, where $x_i(e)$ is the number of occurrences

of e in $X(i)$. Define $B_0(i)$ to be the multiset consisting of $X(i)$ together with $k(n-2m)$ copies of symbol i . We also define $B_0(i)$ for $m+1 \leq i \leq n-m$; in this case we let $B_0(i)$ be the multiset consisting of k copies of each of the symbols $m+1, m+2, \dots, n$. Note that for $1 \leq i \leq m$ and $1 \leq j \leq n-m$, $|A_0(i)| = |B_0(j)| = k(n-m)$, and that each of the symbols from $N(n)$ occurs a total of $k(n-m)$ times in the multisets $A_0(1), \dots, A_0(m), B_0(1), \dots, B_0(n-m)$. The multiset $A_0(i)$ represents the symbols available to complete row i , while $B_0(i)$ helps to ensure completion of the construction but will be discarded.

We now iteratively define a partial k -latin square P_x and multisets $A_x(1), \dots, A_x(m), B_x(1), \dots, B_x(n-m)$, for $1 \leq x \leq k(n-m)$. We assume that for a given p , where $0 \leq p < k(n-m)$, we can choose a symbol from each of the multisets $A_p(1), \dots, A_p(m), B_p(1), \dots, B_p(n-m)$, so that the n symbols chosen are all distinct. We then remove the selected symbols to obtain $A_{p+1}(1), \dots, A_{p+1}(m), B_{p+1}(1), \dots, B_{p+1}(n-m)$. For each $i \in N(m)$, We add the selected symbol from $A_p(i)$ to cell $(i, m+1 + \lfloor p/k \rfloor)$ of P_p ; in this way we obtain P_{p+1} .

We now justify the assumption that for a given p , where $0 \leq p < k(n-m)$, we can choose a symbol from each of the multisets $A_p(1), \dots, A_p(m), B_p(1), \dots, B_p(n-m)$, so that the n symbols chosen are all distinct. By Hall's Theorem, it suffices to show that the union of any s -subset of $\{A_p(1), \dots, A_p(m), B_p(1), \dots, B_p(n-m)\}$ contains at least s distinct symbols. Recall that each of the symbols from $N(n)$ occurs a total of $k(n-m)$ times in the multisets $A_0(1), \dots, A_0(m), B_0(1), \dots, B_0(n-m)$, and that each of these multisets has cardinality $k(n-m)$. At each stage of the iteration we remove one copy of each of the n symbols (one element from each multiset), hence by induction each of the n symbols will occur a total of $k(n-m) - p$ times in the multisets $A_p(1), \dots, A_p(m), B_p(1), \dots, B_p(n-m)$, and each of these multisets will have cardinality $k(n-m) - p$. Thus any s -subset of these multisets will contain a total of $s(k(n-m) - p)$ elements, with each symbol occurring at most $k(n-m) - p$ times, and so must contain at least s distinct symbols.

We have thus shown that the iteration is well defined, and so we obtain a $m \times n$ k -latin rectangle $P_{k(n-m)}$ which contains k elements in each of the cells in the first m rows, and in which the original partial k -latin square P is embedded. Finally, to complete rows $m+1$ through to n , we simply apply Lemma 3.1. \square

In fact, the bound in Theorem 3.1 is the best possible. To see this, observe that any k -latin square of order m cannot be embedded in a k -latin square of order less than $2m$. Since our proof of Theorem 3.1 relies on Lemma 3.1, it does not extend immediately to the case of simple multi-latin squares. However we conjecture the following.

Conjecture 3.1. *For each positive integer k and for each positive integer m , there exists some finite value $n(m, k)$ such that for any $n \geq n(m, k)$, any simple partial k -latin square P of order m embeds in a simple k -latin square of order n .*

4 Erodability and Separability

In this section we explore circumstances under which k -latin squares can be combined or broken down into larger or smaller configurations.

Let L_1 be a k_1 -latin square and L_2 be a k_2 -latin square, each of order n , for some positive integers k_1 and k_2 . Then the *join* of L_1 and L_2 , denoted by $L_1 \oplus L_2$, is the $(k_1 + k_2)$ -latin square L where $L(i, j) = L_1(i, j) \cup L_2(i, j)$ for each $i, j \in N(n)$. Clearly the join is a commutative binary operation.

A k -latin square L is said to be *separable* if there exist $k_1, k_2 < k$ such that $k_1 + k_2 = k$ and L is the join of a k_1 -latin square and a k_2 -latin square; otherwise it is *non-separable*. A k -latin square is said to be *erodable* if it can be expressed as the join of a $(k - 1)$ -latin square and a latin square; otherwise it is *non-erodable*. Note that if a k -latin square is erodable, then it is separable.

However, the converse is not true. For example, in Figure 1, the 4-latin square is non-erodable, but can be written as the join of two 2-latin squares:

$$\begin{array}{|c|c|c|} \hline 1, 1, 2, 2 & 1, 1, 2, 2 & 3, 3, 3, 3 \\ \hline 1, 1, 3, 3 & 2, 2, 3, 3 & 1, 1, 2, 2 \\ \hline 2, 2, 3, 3 & 3, 3, 1, 1 & 1, 1, 2, 2 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 1, 2 & 1, 2 & 3, 3 \\ \hline 1, 3 & 2, 3 & 1, 2 \\ \hline 2, 3 & 3, 1 & 1, 2 \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline 1, 2 & 1, 2 & 3, 3 \\ \hline 1, 3 & 2, 3 & 1, 2 \\ \hline 2, 3 & 3, 1 & 1, 2 \\ \hline \end{array}$$

Figure 1

In fact, it is clear that for any non-erodable k -latin square L with $k \geq 2$, $L \oplus L$ is separable but non-erodable.

A k -latin square is said to be *fully separable* if it can be written as the join of k latin squares.

We next explore the relationship between separability, latin cubes and latin parallelepipeds. An $n \times n \times k$ *latin parallelepiped* is a three dimensional array $A = [a_{i,j,\ell}]$ where $1 \leq i, j \leq n$, $1 \leq \ell \leq k$, $a_{i,j,\ell} \in N(n)$ and $a_{i,j,\ell} \neq a_{i',j',\ell'}$ whenever exactly two of the following conditions hold: $i = i'$, $j = j'$, $\ell = \ell'$. If $k = n$ we say that A is a *latin cube*.

There is a clear equivalence between $n \times n \times k$ *latin parallelepipeds* and simple, fully separable k -latin squares of order n . It is shown in [12] (with further examples in [20]), that not every latin parallelepiped of order n can be extended to a latin cube of order n .

It is an intriguing open problem to determine, for each n , the smallest k such that every $n \times n \times k$ latin parallelepiped may be extended to a latin cube of order n . The best-known constructions come from Kochol:

Theorem 4.1. ([12, 13]) *For each pair of integers d and n such that $d \geq 2$ and $n \geq 2d + 1$, there exists an $n \times n \times (n - d)$ latin parallelepiped that cannot be extended to a latin cube of order n .*

By collapsing each $n \times n \times (n - d)$ latin parallelepiped from the previous theorem into a $(n - d)$ -latin square of order n , then taking the complement (with respect to $N(n)$) of each set in each cell, we obtain the following corollary.

Corollary 4.1. *For all k and n such that $k \geq 2$ and $n \geq 2k + 1$, there exists a k -latin square of order n which is not fully separable.*

The k -latin squares implied by Kochol's construction are, in general, separable and erodable. We will now work towards showing the existence of non-separable k -latin squares of order n , where either $k = 2$ and $n \geq 3$, or when $k \geq 3$ and $n \geq k + 2$.

Observe that a 2-latin square is erodable if and only if it is separable.

Theorem 4.2. *There exists a non-separable 2-latin square of order n if and only if $n \geq 3$.*

Proof. By inspection, each 2-latin square of order 1 or 2 is separable. For each $n \geq 3$, define a 2-latin square L_n of order n as follows.

$$\begin{aligned} L_n(i, 1) &= L_n(i, 2) = \{i, i + 1\}, 1 \leq i \leq n - 2, \\ L_n(i, j) &= \{(i + j - 1 \bmod n, i + j - 1 \bmod n)\}, 1 \leq i \leq n - 2, 3 \leq j \leq n, \\ L_n(n - 1, 1) &= L_n(n, 2) = \{1, n\}, \\ L_n(n, 1) &= L_n(n - 1, 2) = \{n - 1, n\}, \\ L_n(n - 1, j) &= L_n(n, j) = \{j - 2, j - 1\}, 3 \leq j \leq n. \end{aligned}$$

(The 2-latin square L_3 is in Figure 1 and L_4 is in the Introduction.) It is not hard to check that L_n is non-separable for each $n \geq 2$. It is sufficient to consider the first two columns of L_n . \square

Next suppose that $k \geq 3$. We will first construct a non-separable k -latin square of every order from $k + 2$ to $2k + 1$. For ease of understanding we first give the construction for order $k + 2$. Let (K, \odot) be an idempotent quasigroup of order k (where, without loss of generality, $K = N(k)$), and for $a, b \in K$ define $a \odot b = (a + b \bmod k)$. For ease of notation, we use the abbreviation x^y when x occurs y times in a multiset. Define U_k to be a k -latin square of order $k + 2$ with

$$\begin{aligned} U_k(i, i) &= \{i, (k + 1)^{k-1}\}, \quad i \in K, \\ U_k(i \odot 1, i) &= \{k + 1, ((i \odot 1) \odot i)^{k-1}\}, \quad i \in K, \\ U_k(i, j) &= \{k + 2, (i \odot j)^{k-1}\}, \quad i, j \in K, \quad i \neq j, j \odot 1, \\ U_k(i, k + 1) &= U_k(k + 1, i) = \{i^{k-1}, k + 2\}, \quad i \in K, \\ U_k(i, k + 2) &= U_k(k + 2, i) = \{k + 2\} \cup K \setminus \{i\}, \quad i \in K, \\ U_k(k + 1, k + 1) &= U_k(k + 2, k + 2) = K, \\ U_k(k + 1, k + 2) &= U_k(k + 2, k + 1) = (k + 1)^k. \end{aligned}$$

We illustrate with an example for the case $k = 4$. We use the following idempotent quasigroup of order $k = 4$:

\circ	1	2	3	4
1	1	4	2	3
2	3	2	4	1
3	4	1	3	2
4	2	3	1	4

We obtain

$$U_4 = \begin{array}{|c|c|c|c|c|c|} \hline 5, 5, 5, 1 & 4, 4, 4, 6 & 2, 2, 2, 6 & 3, 3, 3, 5 & 1, 1, 1, 6 & 2, 3, 4, 6 \\ \hline 3, 3, 3, 5 & 5, 5, 5, 2 & 4, 4, 4, 6 & 1, 1, 1, 6 & 2, 2, 2, 6 & 1, 3, 4, 6 \\ \hline 4, 4, 4, 6 & 1, 1, 1, 5 & 5, 5, 5, 3 & 2, 2, 2, 6 & 3, 3, 3, 6 & 1, 2, 4, 6 \\ \hline 2, 2, 2, 6 & 3, 3, 3, 6 & 1, 1, 1, 5 & 5, 5, 5, 4 & 4, 4, 4, 6 & 1, 2, 3, 6 \\ \hline 1, 1, 1, 6 & 2, 2, 2, 6 & 3, 3, 3, 6 & 4, 4, 4, 6 & 1, 2, 3, 4 & 5, 5, 5, 5 \\ \hline 2, 3, 4, 6 & 1, 3, 4, 6 & 1, 2, 4, 6 & 1, 2, 3, 6 & 5, 5, 5, 5 & 1, 2, 3, 4 \\ \hline \end{array}$$

Lemma 4.1. *For any $k \geq 3$, and for any idempotent quasigroup (K, \circ) of order k , U_k is a non-separable k -latin square.*

Proof. We leave the proof that U_k is a k -latin square as an exercise.

To prove that U_k is non-separable, suppose that U_k contains an l -latin square S of order $k+2$, $l \geq 1$. Then we have $x \in S(k+1, k+1)$, for some $x \in K$.

The remaining $k-1$ copies of the symbol x in row $k+1$ of U_k occur in cell $(k+1, x)$, and exactly $l-1$ of these copies must occur in S , so $S(k+1, x) = \{x^{l-1}, k+2\}$. To obtain l copies of symbol x in column x , we must have $S(x, x) = \{x, (k+1)^{l-1}\}$. The only other occurrence of symbol $k+1$ in column x of U_k is in cell $(x \odot 1, x)$, so we have $k+1 \in S(x \odot 1, x)$. The remaining $k-1$ copies of symbol $k+1$ in row $x \odot 1$ of U_k occur in cell $(x \odot 1, x \odot 1)$, so $S(x \odot 1, x \odot 1) = \{(k+1)^{l-1}, x \odot 1\}$. In the same way we have $S(k+1, x \odot 1) = \{(x \odot 1)^{l-1}, k+2\}$, and finally $x \odot 1 \in S(k+1, k+1)$.

It follows inductively that $K \subseteq S(k+1, k+1)$, and hence that $l = k$. \square

We now modify the construction for U_k to obtain non-separable k -latin squares of every order between $k+3$ and $2k+1$. Let $s \in \{1, 2, \dots, k-1\}$ and define $K' = N(s+2)$. We use a second idempotent quasigroup (K', \star) of order $s+2$. Define $U_{k,s}$ to be a k -latin square of order $k+s+2$ with

$$\begin{aligned} U_{k,s}(i, i) &= \{i, (k+1)^{k-1}\}, \quad i \in K, \\ U_{k,s}(i \odot 1, i) &= \{k+1, ((i \odot 1) \circ i)^{k-s-1}, k+3, k+4, \dots, k+s+2\}, \quad i \in K, \\ U_{k,s}(i, j) &= \{k+2, (i \circ j)^{k-s-1}, k+3, k+4, \dots, k+s+2\}, \quad i, j \in K, \quad i \neq j, j \odot 1, \\ U_{k,s}(i, k+1) &= U_{k,s}(k+1, i) = \{i^{k-1}, k+2\}, \quad i \in K, \\ U_{k,s}(i, k+x) &= U_{k,s}(k+x, i) = \{k+x\} \cup K' \setminus \{i\}, \quad i \in K, \quad 2 \leq x \leq s+2, \\ U_{k,s}(k+x, k+y) &= K, \quad x, y \in K', \quad x \star y = 1, \\ U_{k,s}(k+x, k+y) &= (k+1)^k, \quad x, y \in K', \quad x \star y = 2, \\ U_{k,s}(k+x, k+x) &= (k+2)^k, \quad x \in K', \quad x \geq 3, \\ U_{k,s}(k+x, k+y) &= (k+z)^k, \quad x, y, z \in K', \quad x \star y = z \geq 3, \quad x \neq y. \end{aligned}$$

We illustrate with an example where $k = 4$ and $s = 2$. We use the same idempotent quasigroup (K, \circ) as above. Since $s + 2 = k$ we let (K', \star) be the same quasigroup also.

We obtain $U_{4,2} =$

5, 5, 5, 1	4, 7, 8, 6	2, 7, 8, 6	3, 7, 8, 5	1, 1, 1, 6	2, 3, 4, 6	2, 3, 4, 7	2, 3, 4, 8
3, 7, 8, 5	5, 5, 5, 2	4, 7, 8, 6	1, 7, 8, 6	2, 2, 2, 6	1, 3, 4, 6	1, 3, 4, 7	1, 3, 4, 8
4, 7, 8, 6	1, 7, 8, 5	5, 5, 5, 3	2, 7, 8, 6	3, 3, 3, 6	1, 2, 4, 6	1, 2, 4, 7	1, 2, 4, 8
2, 7, 8, 6	3, 7, 8, 6	1, 7, 8, 5	5, 5, 5, 4	4, 4, 4, 6	1, 2, 3, 6	1, 2, 3, 7	1, 2, 3, 8
1, 1, 1, 6	2, 2, 2, 6	3, 3, 3, 6	4, 4, 4, 6	1, 2, 3, 4	8, 8, 8, 8	5, 5, 5, 5	7, 7, 7, 7
2, 3, 4, 6	1, 3, 4, 6	1, 2, 4, 6	1, 2, 3, 6	7, 7, 7, 7	5, 5, 5, 5	8, 8, 8, 8	1, 2, 3, 4
2, 3, 4, 7	1, 3, 4, 7	1, 2, 4, 7	1, 2, 3, 7	8, 8, 8, 8	1, 2, 3, 4	6, 6, 6, 6	5, 5, 5, 5
2, 3, 4, 8	1, 3, 4, 8	1, 2, 4, 8	1, 2, 3, 8	5, 5, 5, 5	7, 7, 7, 7	1, 2, 3, 4	6, 6, 6, 6

Again we leave the proof that $U_{k,s}$ is a k -latin square as an exercise. The proof that $U_{k,s}$ is non-separable is exactly the same as the equivalent proof for U_k , because for any $x \in K$ the occurrences of the symbols x and $k + 1$ in row $k + 1$ and column x are identical in U_k and $U_{k,s}$. Thus we have the following:

Corollary 4.2. *For any $k \geq 3$ and $s \in \{1, 2, \dots, k-1\}$, and for any idempotent quasigroups (K, \circ) of order k and (K', \star) of order $s + 2$, $U_{k,s}$ is a non-separable k -latin square of order $k + s + 2$.*

Using the above corollary and Theorem 3.1 from the previous section, we obtain the following theorem:

Theorem 4.3. *For any integers $k \geq 3$ and $n \geq k + 2$, there exists a non-separable k -latin square of order n .*

Even for small values of k and n the above result is not the best possible. For order 4, we exhibit non-separable k -latin squares for each $k \in \{3, 4, 5, 6\}$. These examples were found by computer and checked by hand.

4, 3, 3	2, 2, 2	4, 1, 3	4, 1, 1	4, 2, 3, 3	4, 2, 2, 2	1, 1, 3, 3	4, 4, 1, 1
1, 1, 3	4, 1, 3	2, 2, 2	4, 4, 3	4, 1, 2, 3	1, 1, 2, 3	4, 2, 3, 3	4, 4, 1, 2
2, 2, 2	4, 1, 1	4, 4, 3	1, 3, 3	1, 1, 1, 2	4, 4, 4, 3	4, 2, 2, 2	1, 3, 3, 3
4, 4, 1	4, 3, 3	1, 1, 3	2, 2, 2	4, 4, 2, 3	1, 1, 3, 3	4, 4, 1, 1	2, 2, 2, 3

4, 2, 2, 3, 3	4, 1, 1, 1, 1	1, 2, 2, 2, 3	4, 4, 4, 3, 3
1, 2, 3, 3, 3	4, 4, 2, 2, 2	4, 4, 4, 3, 3	1, 1, 1, 2, 2
4, 4, 4, 2, 2	1, 2, 2, 3, 3	4, 4, 1, 1, 1	1, 2, 3, 3, 3
4, 1, 1, 1, 1	4, 4, 3, 3, 3	1, 2, 2, 3, 3	4, 4, 2, 2, 2

4, 3, 3, 3, 3, 3	2, 2, 2, 2, 2, 3	1, 1, 1, 1, 1, 1	4, 4, 4, 4, 4, 2
1, 1, 2, 2, 2, 3	4, 1, 1, 1, 1, 2	4, 4, 4, 4, 4, 3	2, 2, 3, 3, 3, 3
1, 1, 1, 2, 2, 2	4, 4, 4, 4, 4, 3	2, 2, 2, 3, 3, 3	4, 1, 1, 1, 3, 3
4, 4, 4, 4, 4, 1	1, 1, 3, 3, 3, 3	4, 2, 2, 2, 3, 3	1, 1, 1, 2, 2, 2

We were unable to find a non-separable 7-latin square of order 4. It seems plausible that such a configuration does not exist; however we were unable to check every possible case by computer.

We show the following.

Theorem 4.4. *For each positive integer n , there exists some finite value $g(n)$ such that for any $k \geq g(n)$, every k -latin square of order n is separable.*

Proof. Suppose that $g(n)$ does not exist for some fixed n . Then there exists an infinite sequence (L_t) of non-separable $k(t)$ -latin squares of order n , where $k(t)$ is strictly increasing. For each $i, j, s \in N(n)$, let $m_t(i, j, s)$ be the number of copies of s in cell $L_t(i, j)$. Since our sequence has infinite length, for fixed $(i, j, s) \in N(n) \times N(n) \times N(n)$, the sequence $(m_t(i, j, s))$ contains a non-decreasing sub-sequence of infinite length. Next, replace (L_t) with one of its infinite subsequences so that $(m_t(i, j, s))$ is non-decreasing. We repeat this process for each $(i, j, s) \in N(n) \times N(n) \times N(n)$, obtaining an infinite subsequence (R_r) . Since $k(t)$ is strictly increasing, all but possibly the first multi-latin square in (R_r) is separable. This is a contradiction, so our theorem is true. \square

It is an open problem to determine $g(n)$ exactly for each $n \geq 1$. It is an easy exercise to show that $g(1) = g(2) = 2$. We conjecture that $g(3) = 3$ (the data in Table 1 in the next section certainly supports this) and that $g(4) = 7$.

5 Computation

We remind the reader of the definition of *main class* and *paratopy* given in Section 2.

We wrote C++ code [9] to enumerate k -latin squares using the method of *canonical augmentation* [10], [17]. The main requirement is a function $\mathcal{C}(K)$ that gives a *canonical label* of a (partial) k -latin square. We require that $\mathcal{C}(K) = \mathcal{C}(K')$ if and only if K and K' are paratopic. For this we generalise a well-known graph representation of a latin square [19]. For a k -latin square K of order n we form a graph G_K with vertex set

$$\{v_1, \dots, v_{kn^2}\} \cup \{r_1, \dots, r_n, c_1, \dots, c_n, s_1, \dots, s_n\} \cup \{R, C, S\}. \quad (1)$$

Without loss of generality we may order the elements that appear in each cell of K , so we may speak of the y th element in a cell, for $1 \leq y \leq k$. If e is the y th symbol in cell (i, j) of K , then G_K has the edges (v_ℓ, r_i) , (v_ℓ, c_j) , and (v_ℓ, s_e)

where $\ell = e(n(i-1) + j - 1) + y$. Further, G_K has the edges (r_i, R) , (c_i, C) , (s_i, S) for $1 \leq i \leq n$. Further, we colour the vertices of G_K according to the partitioning in (1). The **nauty** [18] package provides a *canonical label* $\mathcal{C}(G_K)$ such that $\mathcal{C}(G_K) = \mathcal{C}(G_{K'})$ if and only if $G_K^\gamma = G_{K'}$ for some colour-preserving permutation γ of the vertex labels of G_K . Since γ preserves adjacencies and colour classes, the following lemma easily follows.

Lemma 5.1. *Let K and K' be two k -latin squares of order n . Then $\mathcal{C}(G_K) = \mathcal{C}(G_{K'})$ if and only if K and K' are paratopic.*

We now take our canonical label to be $\mathcal{C}(K) = \mathcal{C}(G_K)$. We begin with a partial k -latin square of order n with a single row filled in, and then proceed by adding one row at a time. For small n, k Table 1 shows the number of main classes of k -latin squares of order n , and how many of these are erodable, separable and simple.

n	k	main classes	erodable	separable	simple
3	1	1	0	0	1
3	2	4	3	3	1
3	3	9	9	9	1
3	4	24	22	24	0
3	5	50	50	50	0
3	6	117	115	117	0
3	7	237	237	237	0
3	8	488	485	488	0
3	9	924	924	924	0
4	1	2	0	0	2
4	2	44	26	26	10
4	3	2424	2181	2181	2
4	4	218632	212942	218198	1
5	1	2	0	0	2
6	1	12	0	0	12

Table 1

6 Applications

In this section we briefly discuss some possible applications of multi-latin squares to the design of statistical experiments.

Suppose that we want to compare n varieties of tomato, n types of compost and n watering schemes, and we have n^2k plots to do this in. If we use a multi-latin square, assigning the varieties to rows, composts to columns and watering schemes to symbols, then each variety occurs k times with each compost, each

variety occurs k times with each watering scheme, and each compost occurs k times with each watering scheme. This is a good design if we can assume that there are no *interactions*, which means that the difference in performance between two varieties of tomato does not depend on the type of compost or the watering scheme, and similarly for two types of compost or two watering schemes. Such a design would be called an orthogonal main-effects factorial design for three n -level treatment factors.

For practical purposes, this wouldn't be done for $k \geq n$, because then it would be better to ensure that all of the potential n^3 combinations occurred at least once.

For a second sort of design, instead of watering schemes, suppose that we are going to use n glasshouses, with nk chambers in each glasshouse. Even if we are not interested in the differences between the glasshouses, using a multi-latin square with the varieties, composts and glasshouses assigned to the rows, columns and symbols in some order gives us a good orthogonal main-effects factorial design for two n -level treatment factors in n blocks of size nk .

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